Exploring Complex Algebra in Graphic Design Concepts

Samantha Laqueenna Ginting - 13523138
Informatics Engineering Study Program
School of Electrical Engineering and Infomatics
Institute of Technology Bandung, Ganesha Street 10 Bandung 40132, Indonesia
lana.ginting@gmail.com, 13523138@std.stei.itb.ac.id

***Abstract*—Concepts in the graphic design field can be understood through the lens of mathematics. Complex algebra provides a way to explain and calculate graphic transformations such as rotation, scaling, translation, shearing, reflection, distortion, and holomorphic dynamics. This approach allow designers to apply precise and efficient calculations, leading to more seamless and accurate transformations in their creative work.**

***Keywords*—Graphic design, transformation, complex numbers, coordinate, object.**

I. Introduction

Graphic design is a form of visual communication. This field is an interdisciplinary branch of art, communication, technology, and mathematics. Among these disciplines, mathematics, more specifically complex algebra (a subfield of linear algebra and geometry), serves an important role in various concepts involved in graphic design.

Complex algebra is a tool in understanding and applying two dimensional geometry transformations. A graphic transformation is a process to change the position, shape, size, and/or orientation of a graphic object. These transformations include rotation, scaling, translation, shearing, reflection, distortion, and holomorphic dynamics. By manipulating objects in a coordinate system, designers are able to achieve seamless, accurate, and specific changes.

Furthermore, apart from geometric transformations, complex algebra can also be applied to computer graphic concepts. This includes image processing, three dimensional geometric transformations, texture mapping, and imaging visual effects.

This research focuses on exploring the role of complex algebra in graphic design. It aims to understand the theoretical concepts behind geometry transformations and how they contribute to the development of digital graphic design.

II. Theoretical Background

By definition, the square of a real number is a positive real number. Before the 17th century, operations involving the square roots of negative numbers poses difficulties. Until in 1777, Leonhard Euler introduced the imaginary unit i to stand in for $\sqrt{-1}$.

Because of Euler's imaginary unit, complex numbers are expressed in the form

$$z = a + ib,$$

where

$$a \text{ is the } \textit{real} \text{ part,}$$
$$b \text{ is the } \textit{imaginary} \text{ part, and}$$
$$i = \sqrt{-1} \longleftrightarrow i^2 = -1.$$

The real or the imaginary part may be zero, therefore the set of real numbers $\mathbb{R}$ is a subset of complex numbers $\mathbb{C}$. The unit i has a following pattern,

$$i^0 = 1,\ i^1 = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ ...$$

To find the norm of a complex number z,

$$|z| = \sqrt{a^2 + b^2}.$$

Complex numbers obey all laws of elementary algebra, such as addition, subtraction, and multiplication. But, divisions in complex numbers

$$\tfrac{z_1}{z_2} = \tfrac{a_1+ib_1}{a_2+ib_2}$$

are handled quite special. In order to dissolve the denominator into a real quantity, the conjugate of the denominator will be multiplied with the numerator and the denominator respectively. Conjugate of a complex number is simply a sign reversal of the imaginary part.

$$\text{If } z = a + ib,$$
$$\text{then the conjugate of } z \text{ is}$$
$$z^* = a - ib.$$

When a complex number is multiplied by its conjugate, it will produce a real quantity

$$zz^* = (a + ib)(a - ib) = a^2 + b^2.$$

Therefore, to solve two complex number divisions is as follows

$$\tfrac{z_1}{z_2} = \tfrac{z_1}{z_2} \cdot \tfrac{z_2^*}{z_2^*} = \tfrac{a_1+ib_1}{a_2+ib_2} \cdot \tfrac{a_2-ib_2}{a_2-ib_2}$$
$$\tfrac{z_1}{z_2} = \tfrac{(a_1a_2+b_1b_2)+i(a_2b_1-a_1b_2)}{a_2+ib_2}.$$

Complex numbers can be represented as points on a two dimensional plane. This plane, which is also called as the *Argand diagram*, uses two orthogonal axes, with the horizontal axis as the real axis and the vertical axis as the imaginary axis.

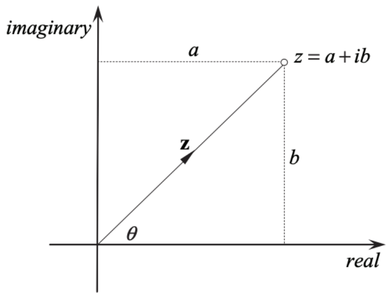

Fig. 1. Argand Diagram

The polar angle θ, which is the angle of z with respect to the real axis, is recognized by using the inverse tangent function $(\tan^{-1})$,

$$\theta = \tan^{-1}\left(\frac{b}{a}\right),$$

where z is in the first and fourth quadrants and $a > 0$. For the second and third quandrants, when $a < 0$,

$$\theta = 180^\circ + \tan^{-1}\left(\frac{b}{a}\right).$$

A complex number can be proportionally rotated with i as its rotor. By multiplying a complex number z by $-i$, z will be rotated clockwise as illustrated below

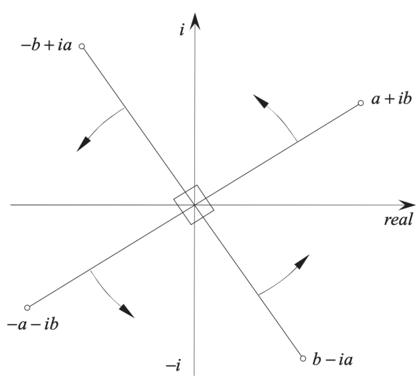

Fig. 2. Complex Number Rotation

From Fig. 1, can be concluded that,

$$\frac{a}{|z|} = \cos\theta, \text{ and}$$
$$\frac{b}{|z|} = \sin\theta.$$

Therefore the complex number z can be written in Cartesian form,

$$z = |z|\cos\theta + i|z|\sin\theta$$
$$z = |z|(\cos\theta + i\sin\theta).$$

Leonhard Euler also discovered a formula obtained from Maclaurin's infinite series. The base of the natural system of logarithms is defined

$$e = \lim_{n\to\infty}(1 + \frac{1}{n})^n,$$

but it can also be represented as an infinite series

$$e = \lim_{n\to\infty}(1 + \frac{1}{1!}) + \frac{1}{2!}) + \frac{1}{3!}) + ... + \frac{1}{n!}).$$

So e^x has the form,

$$e = \lim_{n\to\infty}(1 + \frac{x}{1!}) + \frac{x^2}{2!}) + \frac{x^3}{3!}) + ... + \frac{x^n}{n!}).$$

Similarly, it can be shown that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}...$$
$$\cos x = x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}....$$

With that, the exponentiation of the natural number e with the complex number ix is as follow,

$$e^{ix} = \lim_{n\to\infty}(1 + \frac{ix}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + ... + \frac{i^nx^n}{n!})$$

simplified to

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + ... + \frac{i^nx^n}{n!}.$$

Collecting the real and imaginary terms

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}... + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}...),$$

therefore,

$$e^{ix} = \cos x + i\sin x.$$

Combining from the Cartesian form before, it can be concluded that,

$$z = |z|e^{i\theta}.$$

This is interpreted as the real quantity $|z|$ rotated through an angle θ.

III. Result

Transformation in graphic design is a process to change the position, shape, size, and/or orientation of a graphic object. Graphic transformations are used by designers to create specific visual effects.

Two dimensional shapes or graphic objects are defined by their characteristic points. The characteristic points of polygons are its vertices or corner points. For non-polygon shapes, such as curves, are its control points. To transform the graphic object, apply the same formula to all of its characteristic points z_1, z_2, z_3, ... , z_n. The new shape will be reconstructed by the rotated points.

Here are the different transformations explained within the framework of complex algebra.

1) Rotation

   Rotation is used to change the orientation or direction of a graphic object. In graphic design, rotation allows a graphic object to be rotated at a certain angle.

   Rotation of a point or object can be represented simply using multiplication of complex numbers.

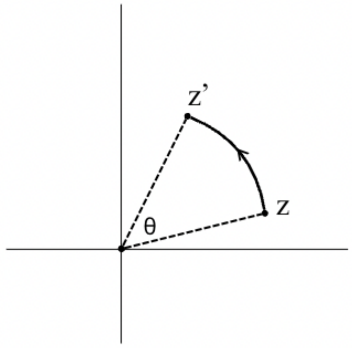

Fig. 3. Rotation

If a complex number $z = x+yi$ is the starting position of a point, then the rotation of z by θ is

$$z' = ze^{i\theta}$$

with $e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's formula).

A point can also be propotionally rotated simply by multiplying $z = a + bi$ with the imaginary unit i. The result of this process is the same as if we rotated z with $\theta = \frac{\pi}{2}$. For example, if we multiply $z = a + bi$ with i,

$$z' = zi = ai + bi^2 = ai - b,$$

and we substitute $\theta = \frac{\pi}{2}$ to the previous formula,

$$z' = ze^{i\frac{\pi}{2}} = (a+bi)(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}) =$$
$$(a+bi)(0+i) = (a+bi)i = ai + bi^2 = ai - b,$$

the same resulted is achieved.

2) Scaling

Scaling is a form of transformation that changes the size of a graphic object by stretching or compressing it. This process involves multiplying a complex number by a scaling factor k, therefore

$$z' = k \cdot z = k(a+bi) = ka + kbi.$$

In two dimensional graphic objects, if $k > 1$, the point will move further away from the origin, making the shape larger. When $0 < k < 1$, the point will be scaled down.

Scaling is used in graphic design to propotionally resize an object along the horizontal and vertical axis.

3) Translation

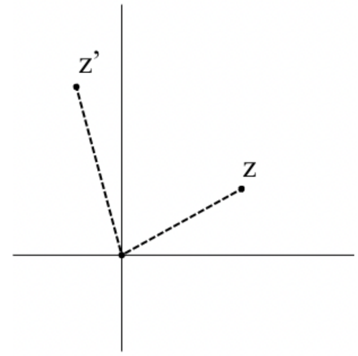

Fig. 4. Translation

Translation is the process of moving an object from one position to another by adding a constant value. Unlike scaling, translation does not alter the size or orientation of the object, it simply moves the object.

For example, if the complex number $z = a + bi$ is translated by $P(x, y)$, then

$$z' = (x + a) + (y + b)i.$$

4) Shearing

Shearing is a transformation where the graphic object's points are shifted in a way that maintains the objects' basic structure but changes its orientation or angle.

There are several types of a two dimensional shearing such as, horizontal shearing (changing the $real$ coordinate), vertical shearing (changing the $imaginary$ coordinate), and combined shearing (changing the $real$ and $imaginary$ coordinates). For example, if a rectangle is sheared horizontally, the new object will be a parallelogram.

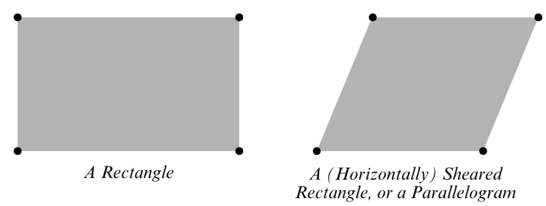

Fig. 5. Shearing A Rectangle

If a complex number is $z = a + bi$ and is going to be sheared with $S(h, v)$, where h is the horizontal shearing factor and v is the vertical shearing factor, then

$$z' = a' + b'i = ((a + h \cdot b) + (b + v \cdot a)i).$$

For horizontal shearing, a' is changed based on its original imaginary component b. This means the shear effect occurs along the $real$ axis, changing how the imaginary part of the complex number behaves. Likewise, for vertical shearing, b' is changed based on its original real component a. For this case, the shear effect occurs along the $imaginary$ axis, changing how the real part of the complex number.

Shearing at a specified angle (θ) can simply be calculated using the rotation operation

$$z' = ze^{i\theta}$$

with $e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's formula).

5) Reflection

Reflection is a form of transformation that mirrors an object over a specified axis of reflection.

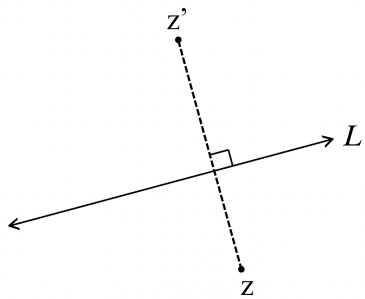

Fig. 6. Reflection

Reflecting a complex number z across the real axis turns z into its conjugate z^*. It is also possible to reflect z through an arbitrary line L with the equation $ax+by=c$, where all of the quantities are real numbers.

In order to reflect $z = x_0 + y_0 i$ through the line L, we need to find a line M that is perpendicular to L and pass through the original point (x_0, y_0). L can be rewritten as,

$$y = -\frac{a}{b}x + \frac{c}{b}$$

and M will be,

$$y - y_0 = \frac{b}{a}x(x - x_0).$$

Then, we determine the intersection point (x_1, y_1) of the two lines,

$$x_1 = -\frac{aby_0 - b^2x_0 - ac}{a^2+b^2}$$

$$y_1 = \frac{a^2y_0 - abx_0 + bc}{a^2+b^2}.$$

The reflection of the complex number z across L involves moving (x_0, y_0) to (x_1, y_1) and then the equal distance to the other side of (x_1, y_1). The formula is as follows,

$$x_2 = x_0 + 2(x_1 - x_0) = 2x_1 - x_0$$

$$y_2 = y_0 + 2(y_1 - y_0) = 2y_1 - y_0.$$

Therefore, the reflection of z through L,

$$z' = x_2 + iy_2.$$

We can reformulate and simplify the search of the reflection point (x_2, y_2) as,

$$x_2 = \frac{-a^2x_0 + (-2by_0 + 2c)a + b^2x_0}{a^2+b^2}$$

$$y_2 = \frac{-b^2y_0 + (-2ax_0 + 2c)b + a^2y_0}{a^2+b^2}.$$

6) Distortion

a) Linear Distortion

Linear distortion is similar to scaling. Scaling is a uniform transformation, where the graphic object is transformed equally along the *real* and *imaginary* axis. Linear distortion enables graphic object to be transformed non-equally along the axes. For example, if the distortion factor is $P(x, y)$, then

$$z' = xa + ybi.$$

The new coordinate will be (xa, yb).

b) Radial Distortion

Radial distortion changes the position of a graphic object based on the radial distance from the center of distortion. This type of distortion is also called as lens distortion.

The formula is as follows,

$$z' = z \cdot (1 + k \cdot r^n)$$

where r is the radial distance from the center of distortion, k is the distortion factor, and n is the exponent to determine the type of distortion.

Based on its distortion factor, radial distortion can be divided into two types. If $k > 0$, the points will be further away from the center, this is called barrel distortion. If $k < 0$, the points will be closer to the center, this is called pincushion distortion.

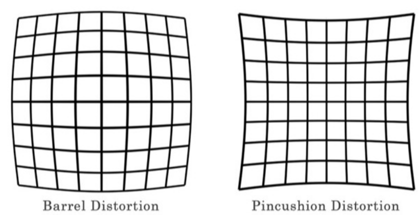

Fig. 7. Barrel and Pincushion Distortion

c) Nonlinear Distortion

Nonlinear distortion is a complex distortion which depends on other functions, such as sinusoidal, exponential, or polinomial functions. The general formula is as follows,

$$z' = f(|z|) \cdot z,$$

where $f(|z|)$ is the function that determines the distortion pattern.

An example of a nonlinear distortion is the ripple effect, where a graphic object is deformed using a sinusoidal pattern.

7) Holomorphic Dynamics

Holomorphic dynamics or complex dynamics is an extensive study of complex transformation obtained by iterating a complex analytic mapping. A holomorphic function has a unique complexity when the iteration is done to the complex number. It can show how the iteration broaden or stretch points until they approach a specific object (like a point or a cycle).

The general formula of a holomorphic function is as follows,

$$f(z) = z^d + c$$

where z is a complex number, d is the degree of function (for example $d = 2$ for quadratic functions), and c is a constant. The iteration function of $f(z)$ is,

$$z_{n+1} = f(z_n) = z_n^2 + c.$$

From that formula, it is shown how the points change over time as the function iterates as shown from the figure below.

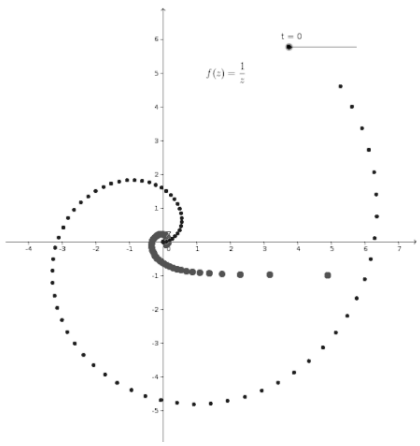

Fig. 8. Mapping of the function $f(z) = \frac{1}{z}$

Holomorphic dynamics has several applications in the graphic design field, especially in producing complex patterns and fractals. One example is fractal art, a type of digital geometric designs that resemble natural patterns.

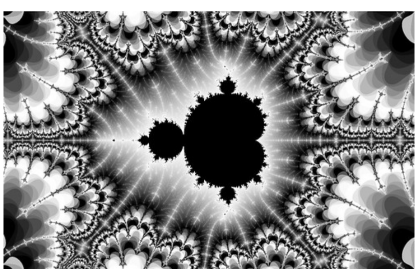

Fig. 9. Fractal Art

IV. Conclusion

Complex algebra is a powerful tool to explain and visualize various transformation concepts in graphic design. Transformation calculations in complex numbers are more time and space efficient than calculating them on a Cartesian coordinate system. This is because of how complex numbers are formalized. For example, to rotate a point (x, y) around the origin point with an angle of θ, we need to use a rotation matrix as follow,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Meanwhile, when represented with a complex number $z = a + bi$, the point's rotation is simply done with the formula,

$$z' = z \cdot e^{i\theta}.$$

With Cartesian coordinate systems, when it is needed to combine a combination of transformations, we need to multiply a couple of matrix, and the result is dependent on its order. These multiple steps are costly, and can be tough to handle numerical errors.

By using complex numbers and algebraic principals, designers can achieve more enhanced graphics and ensure that transformations are applied with more control and flexibility. This mathematical approach not only facilitates better outcomes but also contributes to the development of sophisticated graphics enhancing creativity and functionality in the design process.

V. Common Mistakes

When writing a research paper on graphic transformations linked to mathematic concepts, some mistakes are unavoidable.

Some derivations and steps in the formulas, typically for the more complex graphic transformations are often skipped. This is due to the level of complexity of a particular transformation. Additionally, there have been different derivations in certain formulas throughout history. For example, transformations with holomorphic functions.

To address this problem, readers are advised to explore the references mentioned in the paper. Readers can also conduct further research on the material or have an expert review the mathematical content.

VI. Acknowledgements

First of all, the author would like to thank God for his grace and guidance all throughout the process of writing this paper and learning in this class. The author would also like to thank the lecturers of IF2123 *Aljabar Linier dan Geometri*, Mr. Judhi Santoso and Mr. Arrival Dwi, for sharing their knowledge and insights throughout the learning process in the class.

The author also feels honored to all of the love and support from family members, classmates, and their best friend, Karina. Not only their support, but their work and determination had always inspire the author.

Lastly, the author would like to share their gratitude towards themselves for enduring and always pushing through even when it is hard.

References

[1] Hitchman, Michael P. 2021. *3.1: Basic Transformations of Complex Numbers*. LibreTexts Mathematics. https://math.libretexts.org/Bookshelves/Geometry/Geometry_with_an_Introduction_to_Cosmic_Topology_(Hitchman)/03%3A_Transformations/3.01%3A_Basic_Transformations_of_Complex_Numbers. Accessed on December 29, 2024.

[2] Leys, J., Ghys, É., & Alvarez, A. 2008. *Chapters 5 and 6: Complex Numbers*. https://dimensions-math.org/Dim_CH5_E.htm. Accessed on January 1, 2025.

[3] Munir, Rinaldi. 2024. *Aljabar Kompleks*. https://informatika.stei.itb.ac.id/ rinaldi.munir/AljabarGeometri/2023-2024/Algeo-24-Aljabar-Kompleks-2023.pdf. Accessed on December 25, 2024.

[4] Pettersson, Rune. 2024. *Graphic Design*. Institute for Infology. https://www.researchgate.net/publication/281810469_7_Graphic_Design. Accessed on December 26, 2024.

[5] Read, Jen. 2024. *Lens Distortion: What Every Photographer Should Know*. https://clickitupanotch.com/lens-distortion/. Accessed on January 1, 2025.

[6] Rempe, L. 2008. *An Introduction to Holomorphic Dynamics*. University of Liverpool. https://pcwww.liv.ac.uk/ lrempe/workshops/liv_jan_08/handout_lecture_1.pdf. Accessed on January 1, 2025.

[7] Ross, Sharon. 2009. *Black and White Fractals That Capture Creativity*. Smashing Magazine. https://www.smashingmagazine.com/2009/04/black-and-white-fractals-that-capture-creativity/. Accessed on January 1, 2025.

[8] Rowe, Glenn. 2024. *Reflection of Complex Number Through A Line*. PyhsicsPages. https://physicspages.com/pdf/Mathematics/Reflection%20of%20complex%20number%20through%20a%20line.pdf. Accessed on December 31, 2024.

[9] Vince, John. 2008. *Geometric Algebra for Computer Graphics*. Springer. http://www.drhuang.com/science/mathematics/book/geometric%20algebra%20for%20computer%20graphics.pdf. Accessed on December 27, 2024.

AUTHOR'S STATEMENT

I declare that the paper I wrote is my own writing, not an adaptation, a translation of another paper, and not plagiarism.

Jatinangor – January 2, 2025

Samantha Laqueenna Ginting 13523138